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The Module Factorization of Operators on Hilbert Space

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1. Introduction

It is known that solutions of certain operator equations are closely related to the problem of factoring an operator with respect to a nest $\mathscr{E}$ of projections in a Hilbert space H [3]. A representation $A = ST$ is a factorization of A with respect to the nest $\mathscr{E}$ if S leaves invariant each member of $\mathscr{E}$ and T leaves invariant the orthogonal complement of each member of $\mathscr{E}$ [1]. In infinite dimensions factorizations of this type were first considered by Gohberg and Krein [4]. In this paper we generalize such factorizations of A by introducing the notion of regularity relative to a nest algebra module $\mathscr{U}$ and then use it to classify them. Moreover we give a necessary and sufficient condition, when A is a positive operator, to have a factorization of the form $A = S^*S$ where $S \in \mathscr{U}$ and $S^{-1} \in (\mathscr{U}^{\perp})^*$. This generalizes the main result in [3]. The exposition and structure of this paper follows closely those in [1, 3].

Standard terminology and notation will be used throughout this paper (see, for example, [2]). The terms *Hilbert space*, *operator*, and *projection* will mean complex Hilbert space, bounded linear operator, and otrthogonal projection, respectively. Alg $\mathscr{E}$ denotes the corresponding nest algebra to the nest $\mathscr{E}$. If $E \to \tilde{E}$ is a left continuous order homomorphism of $\mathscr{E}$, let $\mathscr{U} = \{X \in \mathscr{B}(H) : XE = \tilde{E}XE$ for all $E \in \mathscr{E}\}$. $\mathscr{U}$ is an Alg $\mathscr{E}$ module and the set $\mathscr{U}^{\perp} = \{X \in \mathscr{B}(H) : \tilde{E}X = \tilde{E}XE$ for all $E \in \mathscr{E}\}$ is then an Alg $\mathscr{E}^{\perp}$ module. By A^* we mean the adjoint of the operator A and $\mathscr{U}^* = \{A^* : A \in \mathscr{U}\}$.

2. Classification of Module Factorizations

In this section we consider factorizations of an operator A on the Hilbert space H of the form $A = ST$ where $S \in \mathscr{U}$, $S^{-1} \in (\mathscr{U}^{\perp})^*$ and $T \in \mathscr{U}$, $T^{-1} \in \mathscr{U}^*$. First we show

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PROPOSITION 1. *Let $\mathscr{E}$ be a complete nest of projections and $\mathscr{U}$ an* Alg $\mathscr{E}$ *module. Suppose that there exists an invertible operator T such that T and T^{-1} belong to $\mathscr{U}$. Then the order homomorphism $E \to \tilde{E}$ which determines $\mathscr{U}$ is such that $E \leqslant \tilde{E}$ for every $E \in \mathscr{E}$.*

Proof. Since

$$E = T^{-1}TE = T^{-1}\tilde{E}TE = \tilde{\tilde{E}}T^{-1}\tilde{E}TE = \tilde{\tilde{E}}T^{-1}TE = \tilde{\tilde{E}}E,$$

it follows that $\tilde{\tilde{E}} \geqslant E$ for all $E \in \mathscr{E}$ which implies that $\tilde{E} \geqslant E$ for every $E \in \mathscr{E}$.

A consequence of the above Proposition 1 is that if S is an invertible operator such that $S \in \mathscr{U}^{\perp}$ and $S^{-1} \in \mathscr{U}^{\perp}$ then $E \geqslant \tilde{E}$ for every $E \in \mathscr{E}$. This shows that we cannot generalize the notion of the regular factorization of an operator (see [1]) relative to the pair of modules $\mathscr{U}, \mathscr{U}^{\perp}$ because then the homomorphism $E \to \tilde{E}$ becomes the identity and $\mathscr{U} = \operatorname{Alg} \mathscr{E}$. But it is possible to have such a generalization when we use the modules $\mathscr{U}$ and $(\mathscr{U}^{\perp})^*$. The following example [5] justifies this.

EXAMPLE. Let H be the Hilbert space $L_2[0, 1]$ and $\mathscr{E}$ the nest

$$\{E_t \colon t \in [0, 1], \text{ where } E_t \text{ is the projection on the subspace } L_2[0, t]\}.$$

Consider a function $\varphi\colon [0, 1] \to [0, 1]$ which is onto and differentiable with $\frac{1}{2} \leqslant \varphi' \leqslant 2$. Then there exists the inverse φ^{-1} of φ and satisfies $\frac{1}{2} \leqslant (\varphi^{-1})' \leqslant 2$. The correspondence $E_t \to E_{\varphi(t)}$ is an order homomorphism from $\mathscr{E}$ into itself. This homomorphism determines an Alg $\mathscr{E}$ module $\mathscr{U}$. Define the operator

$$A\colon L_2[0, 1] \to L_2[0, 1], \qquad Af = f \circ \varphi^{-1}.$$

It is easy to see that A is bounded and invertible. Moreover $A \in \mathscr{U}$, $A^{-1} \in (\mathscr{U}^{\perp})^*$, $\mathscr{U} \neq \operatorname{Alg} \mathscr{E}$, $\mathscr{U} \neq (\mathscr{U}^{\perp})^*$, and $A \notin \operatorname{Alg} \mathscr{E}$.

DEFINITION 2. A representation

$$A = ST \tag{1}$$

is called a *regular* factorization of A with respect to the module $\mathscr{U}$ if S and T are invertible and $S \in \mathscr{U}$, $S^{-1} \in (\mathscr{U}^{\perp})^*$, $T \in \mathscr{U}^{\perp}$, and $T^{-1} \in \mathscr{U}^*$. The representation (1) is called a *left regular* factorization of A with respect to the module $\mathscr{U}$ if S is invertible, $S \in \mathscr{U}$, $S^{-1} \in (\mathscr{U}^{\perp})^*$, and $T \in \mathscr{U}^{\perp}$. A *right regular* factorization is defined similarly.

In the following lemma we prove that a regular factorization is unique up to a multiplicative factor from the algebra $\operatorname{Alg} \mathscr{E} \cap \operatorname{Alg} \tilde{\mathscr{E}}^{\perp}$.

LEMMA 3. *If $A = S_1 T_1$ and $A = S_2 T_2$ are two regular factorizations of A with respect to the module $\mathscr{U}$ then there exists an operator D in $\operatorname{Alg} \mathscr{E} \cap \operatorname{Alg} \tilde{\mathscr{E}}^{\perp}$ such that $S_1 = S_2 D$ and $T_2 = D T_1$.*

Proof. Let $A = S_1 T_1 = S_2 T_2$. Then

$$S_2^{-1} S_1 = T_2 T_1^{-1}. \tag{2}$$

Since the factorizations are regular we have

(i) $S_2^{-1} S_1 E = S_2^{-1} \tilde{E} S_1 E = E S_2^{-1} \tilde{E} S_1 E = E S_2^{-1} S_1 E$ for every $E \in \mathscr{E}$ and so $S_2^{-1} S_1 \in \operatorname{Alg} \mathscr{E}$.

(ii) $\tilde{E} T_2 T_1^{-1} = \tilde{E} T_2 E T_1^{-1} = \tilde{E} T_2 E T_1^{-1} \tilde{E} = \tilde{E} T_2 T_1^{-1} \tilde{E}$ for every $E \in \mathscr{E}$ and so $T_2 T_1^{-1} \in \operatorname{Alg} \tilde{\mathscr{E}}^{\perp}$.

Therefore from (2) and (i), (ii) we have that $S_2^{-1} S_1$, $T_2 T_1^{-1}$ belong to $\operatorname{Alg} \mathscr{E} \cap \operatorname{Alg} \tilde{\mathscr{E}}^{\perp}$ and hence there exists an operator $D \in \operatorname{Alg} \mathscr{E} \cap \operatorname{Alg} \tilde{\mathscr{E}}^{\perp}$ such that $S_2^{-1} S_1 = T_2 T_1^{-1} = D$, and the proof is completed.

In the sequel we examine when a left regular factorization is regular.

LEMMA 4. *Suppose T is an invertible operator in $\mathscr{B}(H)$. Then if $T \in \mathscr{U}$ the following are equivalent.*

(i) $T^{-1} \in (\mathscr{U}^{\perp})^*$.

(ii) $(\tilde{E} - \tilde{F})\, T(E - F)$ $(\tilde{E} - \tilde{F} \neq 0,\ E > F)$ *is invertible for all* $E, F \in \mathscr{E}$.

(iii) $\tilde{E} T E$ $(\tilde{E} \neq 0)$ *is invertible for all* $E \in \mathscr{E}$.

(iv) $(I - \tilde{E})\, T(I - E)$ $(\tilde{E}, E \neq I)$ *is invertible for all* $E \in \mathscr{E}$.

Proof. Suppose (i) holds. Then for E, $F \in \mathscr{E}$, $E > F$, $\tilde{E} \neq \tilde{F}$ we have $\tilde{E} > \tilde{F}$ and since $T^{-1} \tilde{E} = E T^{-1} \tilde{E}$, $(I - F)\, T^{-1} = (I - F)\, T^{-1}(I - \tilde{F})$, and $(I - \tilde{F}) T = (I - \tilde{F})\, T(I - F)$ we have

$$\begin{aligned}
(\tilde{E} - \tilde{F})\, T(E - F)(E - F)\, T^{-1}(\tilde{E} - \tilde{F}) &= (\tilde{E} - \tilde{F})\, T(E - F)\, T^{-1}(\tilde{E} - \tilde{F}) \\
&= (I - \tilde{F})\, \tilde{E} T E(I - F)\, T^{-1}(I - \tilde{F}) \tilde{E} \\
&= (I - \tilde{F})\, T E(I - F)\, T^{-1} \tilde{E} \\
&= (I - \tilde{F})\, T T^{-1} \tilde{E} \\
&= \tilde{E} - \tilde{F}.
\end{aligned}$$

Similarly $(E-F)\,T^{-1}(\tilde{E}-\tilde{F})(\tilde{E}-\tilde{F})\,T(E-F)=E-F$ and hence (ii) follows. Clearly, since $\tilde{0}=0$, (ii) implies (iii), and we can easily prove that (i) implies (iv). Suppose (iii) holds. Then

$$\tilde{E}TET^{-1}\tilde{E}=TET^{-1}\tilde{E}=\tilde{E} \qquad \text{for every} \quad E\in\mathscr{E},\ \tilde{E}\neq 0$$

and

$$\begin{aligned}
T[ET^{-1}\tilde{E}&+(I-E)\,T^{-1}(I-\tilde{E})+ET^{-1}(I-\tilde{E})]\\
&=TET^{-1}\tilde{E}+T(I-E)\,T^{-1}(I-\tilde{E})+TET^{-1}(I-\tilde{E})\\
&=TET^{-1}\tilde{E}+I-\tilde{E}-TET^{-1}+TET^{-1}\tilde{E}+TET^{-1}-TET^{-1}\tilde{E}=I.
\end{aligned}$$

Therefore

$$T^{-1}=ET^{-1}\tilde{E}+(I-E)\,T^{-1}(I-\tilde{E})+ET^{-1}(I-\tilde{E})$$

and so

$$T^{-1}\tilde{E}=ET^{-1}\tilde{E} \qquad \text{for every} \quad E\in\mathscr{E},\ \tilde{E}\neq 0.$$

But since the only projection $E\in\mathscr{E}$ satisfying $\tilde{E}=0$ is the projection $E=0$ ($TE=\tilde{E}TE=0TE=0$ implies $T^{-1}TE=0$ and hence $E=0$) we have $T^{-1}\tilde{E}=ET^{-1}\tilde{E}$ for all $E\in\mathscr{E}$ and hence $T^{-1}\in(\mathscr{U}^{\perp})^*$. Finally with the same argument as above we prove that (iv) implies (i).

PROPOSITION 5. *Suppose $A=ST$ is a left regular factorization of A with respect to the module $\mathscr{U}$. Then the following are equivalent*:

(i) *$A=ST$ is regular.*

(ii) *$EA^{-1}\tilde{\tilde{E}}$ $(E,\tilde{\tilde{E}}\neq 0)$ is invertible for every $E\in\mathscr{E}$.*

(iii) *$(I-\tilde{\tilde{E}})\,A(I-E)$ $(E,\tilde{\tilde{E}}\neq I)$ is invertible for every $E\in\mathscr{E}$.*

Proof. Let $A=ST$ be a regular factorization. Then Lemma 3 shows that for every $E\in\mathscr{E}$, $\tilde{\tilde{E}}\neq 0$ the operators $ET^{-1}\tilde{E}$, $\tilde{E}S^{-1}\tilde{\tilde{E}}$ are invertible and for every $E\in\mathscr{E}$, $\tilde{E}$, $\tilde{\tilde{E}}\neq I$ the operators $(I-\tilde{\tilde{E}})\,S(I-\tilde{E})$, $(I-\tilde{E})\,T(I-E)$ are also invertible. Therefore from

$$EA^{-1}\tilde{\tilde{E}}=ET^{-1}S^{-1}\tilde{\tilde{E}}=(ET^{-1}\tilde{E})(\tilde{E}S^{-1}\tilde{\tilde{E}})$$

and

$$\begin{aligned}
(I-\tilde{\tilde{E}})\,A(I-E)&=(I-\tilde{\tilde{E}})\,ST(I-E)\\
&=(I-\tilde{\tilde{E}})\,S(I-\tilde{E})\,T(I-E)\\
&=(I-\tilde{\tilde{E}})\,S(I-\tilde{E})(I-\tilde{E})\,T(I-E)
\end{aligned}$$

we have the invertibility of $EA^{-1}\tilde{\tilde{E}}$ and $(I-\tilde{\tilde{E}})\,A(I-E)$. Suppose (ii) holds. Then

$$(\tilde{E}TE)(EA^{-1}\tilde{\tilde{E}}) = \tilde{E}TA^{-1}\tilde{\tilde{E}} = \tilde{E}S^{-1}\tilde{\tilde{E}}$$

and since the operators $EA^{-1}\tilde{\tilde{E}}$, $\tilde{E}S^{-1}\tilde{\tilde{E}}$ are invertible we have that $\tilde{E}TE$ is invertible, and so, from Lemma 3, $T^{-1} \in \mathscr{U}^*$. Therefore the factorization is regular. Similarly (iii) implies (i).

COROLLARY 6. *If $A = ST$ is a regular factorization with respect to the module $\mathscr{U}$ the operators $I - E + EA^{-1}\tilde{\tilde{E}}$ and $\tilde{\tilde{E}} + (I-\tilde{\tilde{E}})\,A(I-E)$ are invertible.*

Proof. Since from Proposition 4 the inverses of $EA^{-1}\tilde{\tilde{E}}$ and $(I-\tilde{\tilde{E}})\,A(I-E)$ exist we can easily prove that the inverse of $I-E+EA^{-1}\tilde{\tilde{E}}$ is the operator $I-\tilde{\tilde{E}}+\tilde{\tilde{E}}AE$ and the inverse of $\tilde{\tilde{E}}+(I-\tilde{\tilde{E}})\,A(I-E)$ is the operator $E+(I-E)\,A^{-1}(I-\tilde{\tilde{E}})$.

3. THE MODULE FACTORIZATION OF POSITIVE OPERATORS

In this section we give a necessary and sufficient condition for a positive operator A to have a factorization of the form $A = S^*S$ with $S \in \mathscr{U}$ and $S^{-1} \in (\mathscr{U}^{\perp})^*$. This is a generalization of the main result in [3].

Let $\mathscr{N}$ be a nest of subspaces of the Hilbert space H, $\mathscr{E} = \{E_N : N \in \mathscr{N}\}$ the corresponding nest of projections, and $\mathscr{U}$ an Alg $\mathscr{E}$ module determined by the order homomorphism $E_N \to \tilde{E}_N$.

THEOREM 7. *Let A be a positive operator on H. Then $A = S^*S$ with $S \in \mathscr{U}$, $S^{-1} \in (\mathscr{U}^{\perp})^*$ if and only if there exists a unitary operator U such that $U\tilde{E}_N = E_{A^{1/2}N}U$ for every $E_N \in \mathscr{E}$.*

Proof. Suppose that there exists a unitary operator U such that $U\tilde{E}_N = E_{A^{1/2}N}U$. Let $S = U^*A^{1/2}$ and $\tilde{N}$ be the range of $\tilde{E}_N$. Then $A = S^*S$ and $S(N) = SE_N(H) = U^*A^{1/2}E_N(H) = \tilde{E}_N U^*(H) = \tilde{E}_N(H) = \tilde{N}$ for every $N \in \mathscr{N}$. Therefore, from $S(N) = \tilde{N}$, we have $\tilde{E}_N SE_N = SE_N$ and $E_N S^{-1}\tilde{E}_N = S^{-1}\tilde{E}_N$ for every $N \in \mathscr{N}$. Hence $S \in \mathscr{U}$ and $S^{-1} \in (\mathscr{U}^{\perp})^*$. Conversely, let $A = S^*S$ with $S \in \mathscr{U}$ and $S^{-1} \in (\mathscr{U}^{\perp})^*$. It is proved in [3, Corollary 3] that the orthogonal projection on $A^{1/2}E_N(H)$ is $A^{1/2}(E_N A E_N)^{-1}A^{1/2}$. But

$$\begin{aligned} A^{1/2}(E_N A E_N)^{-1}A^{1/2} &= A^{1/2}(E_N S^* S E_N)^{-1}A^{1/2} \\ &= A^{1/2}[(E_N S^* \tilde{E}_N)(\tilde{E}_N S E_N)]^{-1}A^{1/2}. \qquad (3) \end{aligned}$$

From the fact that $S^{-1} \in (\mathscr{U}^{\perp})^*$, Lemma 3 shows that the operators $\tilde{E}_N S E_N$ and $E_N S^* \tilde{E}_N$ are invertible, and hence from (3) we have

$$\begin{aligned} A^{1/2}(E_N A E_N)^{-1} A^{1/2} &= A^{1/2}[(E_N S^{-1} \tilde{E}_N)(\tilde{E}_N S^{*-1} E_N)] A^{1/2} \\ &= A^{-1/2} A [(E_N S^{-1} \tilde{E}_N)(\tilde{E}_N S^{*-1} E_N)] A A^{-1/2} \\ &= A^{-1/2} S^* [(S E_N S^{-1} \tilde{E}_N)(\tilde{E}_N S^{*-1} E_N S^*)] S A^{-1/2} \\ &= A^{-1/2} S^* [(S S^{-1} \tilde{E}_N)(\tilde{E}_N S^{*-1} S^*) S] A^{-1/2} \\ &= A^{-1/2} S^* \tilde{E}_N S A^{-1/2}. \end{aligned}$$

Now if we put $U = A^{-1/2} S^*$ then U is a unitary operator and we have $E_{A^{1/2}N} = U \tilde{E}_N U^*$ or equivalently $U \tilde{E}_N = E_{A^{1/2}N} U$.

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